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LETTER TO THE EDITOR

The ferro/antiferromagnetic q-state Potts model

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Abstract

The critical properties of the mixed ferro/antiferromagnetic q-state Potts model on the square lattice are investigated using the numerical transfer matrix technique. The transition temperature is found to be substantially lower than previously found for $q = 3$. It is conjectured that there is no transition for $q > 3$, in contradiction with previous results.

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In this letter we investigate the anisotropic q-state Potts model on the square lattice [1,2]. This model is defined by the Hamiltonian

$$\mathcal{H} = -\sum_{x,y}(J_x\delta(\sigma_{x,y} - \sigma_{x+1,y}) + J_y\delta(\sigma_{x,y} - \sigma_{x,y+1})) \tag{1}$$

where the variables $\sigma_{x,y}$ may take one of q distinct values and (x, y) runs over the coordinates of the sites of the lattice.

Much is known in the cases when J_x and J_y are of the same sign [2, 3], corresponding to the ferromagnetic Potts model ($J_x > 0$, $J_y > 0$) and the antiferromagnetic Potts model ($J_x < 0$, $J_y < 0$) [4]. Much less is known in the mixed interaction case, $J_x > 0$ and $J_y < 0$. This will be the object of this letter.

There have been a number of attempts to study the mixed interaction model, particularly for the case $q = 3$ [5–8]. Kinzel *et al* [5] obtained a candidate critical line by applying a one-step Migdal–Kadanoff procedure, thereby mapping the model onto the ferromagnetic Potts model for which the exact expression for the critical line is known. This gave the critical line as

$$(1 + \exp(\beta J_x))(1 - \exp(\beta J_y)) = q. \tag{2}$$

Spurred on by the fact that this line is exact for the exactly known $q = 2$ case, it was suggested that perhaps this equation may remain exact for other values of q. Kinzel *et al* also gave numerical results from Monte Carlo simulations which suggest a transition line at lower temperatures than given by (2). The transition line was estimated, however, by extrapolating the inverse correlation lengths by hand to zero for lattice sizes less than about 60.

The possibility that (2) may correspond to an exact result was refuted by the observation that, except for $q = 2$, this line does not remain invariant under the symmetries of the model [6]. The symmetries of the model may be used to construct a symmetry group [9]. The critical line must remain invariant under the action of these symmetries. Using the group of symmetries of the model, Truong [6] constructed extended duality relations which he used to look for candidate critical lines. For the 3-state Potts model this gave

$$\exp(\beta(J_x + J_y)) + 2\exp(\beta J_x) - \exp(\beta J_y) + 1 = 0. \tag{3}$$

Interestingly, this line has the same functional form as the ferromagnetic transition line but with $J_y \to -J_y$. This line is indeed invariant under all the symmetries of the model, but does not seem to correspond to the transition line when compared with numerical results of Kinzel *et al* [5, 7], Yasumura [8] and of the present work. This leaves the interesting question: does this very special line have a corresponding physical meaning? This question remains open.

It has been conjectured that the transition for $q = 3$ is of Kosterlitz–Thouless type. This conjecture is based on the free-fermion approximation applied to a related clock model [10], and supported by numerical results on a related one-dimensional quantum Potts model [11]. In this letter we present convincing evidence that the transition for $q = 3$ is indeed of the Kosterlitz–Thouless type. The critical exponent ν is given as a function of q and shown to diverge at $q = 3$. The critical temperatures are also given, as a function of q, for the case $J_x = -J_y > 0$. No transition is found for $q > 3$. This work provides a more accurate estimate of the critical line for $q = 3$, which is substantially lower than previous estimates [5, 7, 8].

The numerical results were obtained using the numerical transfer matrix method coupled with phenomenological renormalization [12]. The method consists of expressing the partition function of an infinite strip of finite width L in terms of a matrix product. Imposing periodic boundary conditions in the transfer direction (along the direction taken to be infinite), the partition function for the strip may be written

$$\mathcal{Z}_L = \lim_{N\to\infty} \mathrm{Tr}\, T^N. \tag{4}$$

The thermodynamic limit corresponds to the limit $L \to \infty$. For integer values of $q > 1$ it is possible to write the transfer matrices directly in terms of spins. Full details are given by Foster *et al* [13]. For non-integer values of q it is necessary to use the Kasteleyn–Fortuin mapping [14], which gives the Potts model in terms of bond clusters on the lattice. Full details on the construction of the transfer matrices for this case are given in Blöte and Nightingale [15].

It may be shown that the correlation length in the transfer direction is given by

$$\xi_L^{-1} = \log\left(\frac{\lambda_0}{\lambda_1}\right). \tag{5}$$

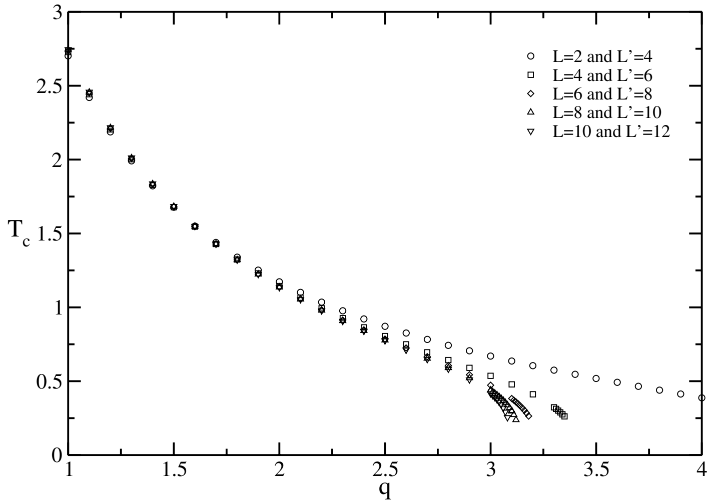
Assuming scale invariance at the critical point, finite-size estimates of the critical temperature may be identified with solutions of the equation

$$\frac{\xi_L(T^*_{L,L'})}{L} = \frac{\xi_{L'}(T^*_{L,L'})}{L'}. \tag{6}$$

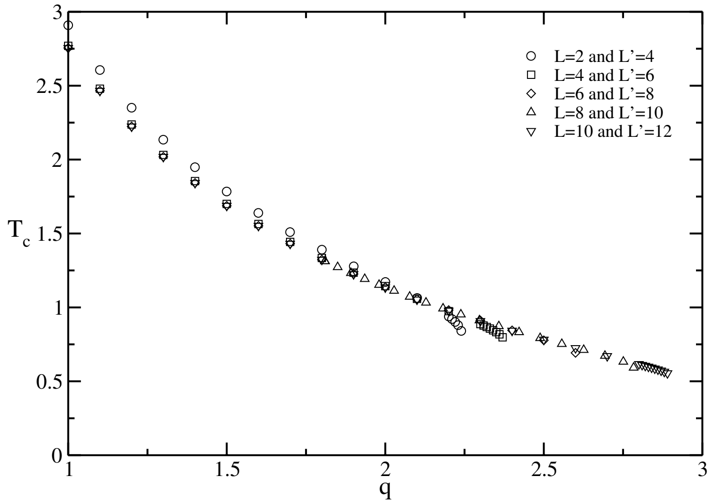
The critical exponent ν may be estimated at the fixed points, solutions of (6), by

$$\frac{1}{\nu_{L,L'}} = \frac{\log\left(\frac{\mathrm{d}\xi_L}{\mathrm{d}T}\Big/\frac{\mathrm{d}\xi_{L'}}{\mathrm{d}T}\right)}{\log\left(\frac{L}{L'}\right)} - 1. \tag{7}$$

Applying these results to the case $J_x = -J_y > 0$ gives the results shown in figure 1 for the critical temperature as a function of q. In figure 1(a) the transfer direction is taken in the x-direction, i.e. parallel to the ferromagnetic interactions, and in figure 1(b) the transfer



(*a*)



(*b*)

Figure 1. The critical temperature as a function of q for a transfer direction along (*a*) the x-direction and (*b*) the y-direction. Periodic boundary conditions are taken in both directions.

direction is taken in the y-direction, i.e. parallel to the antiferromagnetic interactions. In the first case the largest value of q for which a solution was found, $q_{\max}(L)$, decreases with L. On the other hand, $q_{\max}(L)$ increases with L in the second. This indicates a value of q, $q_{\max}$, such that there exists a critical phase transition for $q \leqslant q_{\max}$, but not for $q > q_{\max}$. Figure 2 shows the values of $q_{\max}$ found in the two cases. The curves both extrapolate plausibly to $q_{\max} = 3$ in the thermodynamic limit. Using the upper line in figure 2 to extrapolate, we give $q_{\max} = 3.00 \pm 0.003$. This is in contradiction with Yasumura [8] who gives non-zero values of T_c even for $q > 3$.

The finite-size estimates of ν are shown in figure 3 for a transfer direction in the x-direction. They converge well to $\nu = 1$ for $q = 2$, and diverge for $q = 3$. Table 1 shows the finite-size

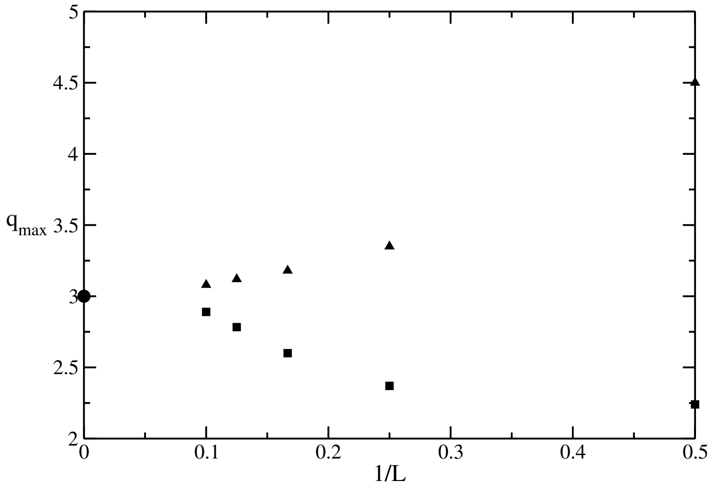


Figure 2. The values of $q_{\max}$, the largest values of q for which an estimate of a critical temperature could be found, as a function of $1/L$. The upper line corresponds to the transfer direction in the x-direction, while the lower line of points corresponds to the transfer direction in the y-direction. The point $q = 3$ and $1/L = 0$ is shown for reference.

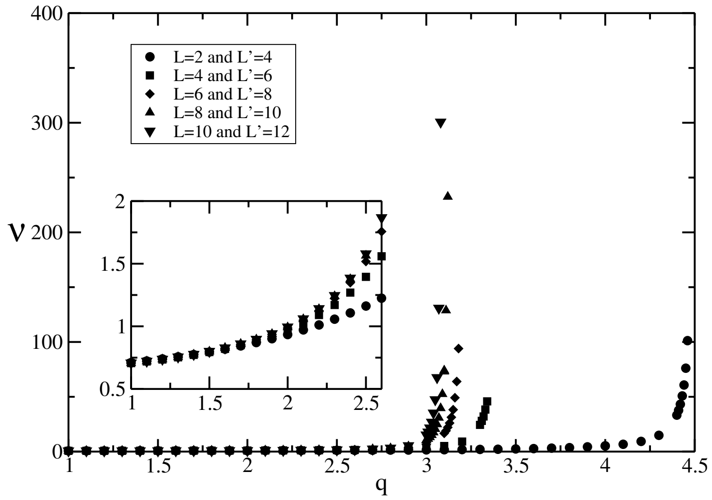


Figure 3. Finite-size estimates of ν as a function of q.

estimates of T_c for selected values of q. The estimates of T_c are extrapolated using a Burlich and Stoer algorithm for $q = 1$, 1.5, 2 and 2.8, but had to be extrapolated graphically for $q = 3$.

The phase diagram for $q = 3$ is shown in figure 4. The critical line estimates obtained are compared with the exact line for $J_y/J_x > 0$, and are shown to converge well, and are compared with both proposed critical lines. The convergence is much slower for $J_y/J_x < 0$, but clearly rules out both proposed analytic results [5, 6] and the numerical results [5, 7, 8].

In summary, in this letter we have addressed numerically the nature of the criticality in the only sector of the anisotropic Potts model for which exact solutions do not as yet exist: the ferro/antiferromagnetic Potts model. We have given critical temperature estimates and estimates of ν as a function q for $J_y/J_x = -1$. We find convincing evidence that there exists

Table 1. Estimates of T_c for selected values of q. The transfer direction is taken in the x-direction.

L/L'	$q = 1$	$q = 1.5$	$q = 2$	$q = 2.8$	$q = 3$
2/4	2.701 840	1.677 124	1.172 009	0.742 794	0.670 151
4/6	2.728 013	1.678 307	1.143 901	0.642 965	0.536 430
6/8	2.737 827	1.680 590	1.137 474	0.606 052	0.472 832
8/10	2.742 918	1.682 553	1.135 844	0.590 809	0.438 814
10/12	2.745 847	1.683 806	1.135 255	0.583 354	0.416 617
∞	2.75 ± 0.01	1.69 ± 0.01	1.13 ± 0.01	0.57 ± 0.01	0.32 ± 0.03

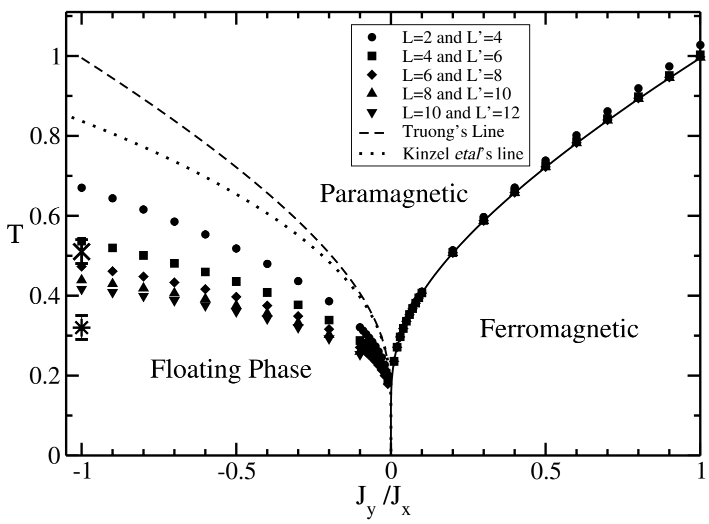


Figure 4. Phase diagram for $q = 3$. The cross shows the estimated critical temperature of Kinzel *et al* [5] for $J_y/J_x = -1$, along with error bar. The star shows the extrapolation of our estimated critical temperatures, also for $J_y/J_x = -1$, with error bar.

a $q_{\max}$ around $q = 3$ above which there is no transition. At $q_{\max}$ the value of ν is diverging, consistent with a Kosterlitz–Thouless type transition. This type of transition has already been conjectured for $q = 3$ using the free-fermion approximation [10]. We conjecture, based on this and our numerical data, that $q_{\max} = 3$. This picture is analogous with what happens in the $O(n)$ spin models in two dimensions, where there is a transition for $-2 \leqslant n \leqslant 2$, the transition at $n = 2$ corresponding to the Kosterlitz–Thouless transition [17].

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References

[1] Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106
[2] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235
[3] Baxter R G 1970 *J. Math. Phys.* **11** 784
[4] Baxter R G 1982 *Proc. R. Soc.* A **383** 42
[5] Kinzel W, Selke W and Wu F Y 1981 *J. Phys. A: Math. Gen.* **14** L399
[6] Truong T T 1984 *J. Phys. A: Math. Gen.* **17** L473
[7] Selke W and Wu F Y 1987 *J. Phys. A: Math. Gen.* **20** 703
[8] Yasumura K 1987 *J. Phys. A: Math. Gen.* **20** 4975

[9] Jaekel M T and Maillard J M 1982 *J. Phys. A: Math. Gen.* **15** 2241
Maillard J M and Rammal R 1983 *J. Phys. A: Math. Gen.* **16** 353
[10] Oustland S 1981 *Phys. Rev.* B **24** 398
[11] Herrmann H J and Martin H O 1984 *J. Phys. A: Math. Gen.* **17** 657
[12] Nightingale M P 1976 *Physica* A **83** 561
[13] Foster D P, Gérard C and Puha I 2001 *J. Phys. A: Math. Gen.* **34** 5183
[14] Kasteleyn P W and Fortuin C M 1969 *J. Phys. Soc. Japan* (suppl) **46** 11
[15] Blöte H W J and Nightingale M P 1982 *Physica* A **112** 405
[16] Henkel M and Schütz 1988 *J. Phys. A: Math. Gen.* **21** 2617
[17] Kosterlitz J M and Thouless F J 1973 *J. Phys. C: Solid State Phys.* **6** 1181